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Comment on "Phase synchronization in discrete chaotic systems"

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Chen *et al.* [Phys. Rev. E **61**, 2559 (2000)] recently proposed an extension of the concept of phase for discrete chaotic systems. Using the newly introduced definition of phase they studied the dynamics of coupled map lattices and compared these dynamics with phase synchronization of coupled continuous-time chaotic systems. In this paper we illustrate by two simple counterexamples that the angle variable introduced by Chen *et al.* fails to satisfy the basic requirements to the proper phase. Furthermore, we argue that an extension of the notion of phase synchronization to generic discrete maps is doubtful.

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The aim of the recent paper by Chen *et al.* [1] was to extend the notion of phase synchronization for the case of coupled chaotic maps. An obvious requirement to any extension of a previously introduced notion is that it must cover the old case. We claim that the angle variable introduced by Chen *et al.* [1] does not satisfy this requirement and illustrate it with two counterexamples. So, the definition of the phase of a chaotic continuous-time self-sustained system proposed in Refs. [2,3] provides the correct phase $\varphi=\omega t$ for periodic (limit cycle) oscillators [4]; that is not the case if the algorithm [1] is used.

Counterexample 1. Consider a closed curve in the phase plane of a two-dimensional map (Fig. 1); a revolution around the curve should provide a 2π increase in phase. Suppose that the motion has (discrete) period 5. Let us compute the angle variable as proposed in Ref. [1]. The procedure consists of two steps. First one computes for each point of the trajectory the angle ϕ_i between the vector that is drawn to the $(i+1)$th point and the x axis. Next, this variable should be recalculated to assure the condition that the "phase" ψ_i is monotonically increasing. This is done according to the following rule: for the first point $\psi_1=\phi_1$; ψ_2 is taken as ϕ_2 if $\phi_2>\phi_1$, and as $\phi_2+2\pi$ if $\phi_2\leqslant\phi_1$. For the next points, $\psi_{i+1}=\phi_{i+1}+2\pi\cdot m$ if $\phi_{i+1}>\phi_i$, and $\psi_{i+1}=\phi_{i+1}+2\pi\cdot(m+1)$, otherwise; here m is the current number of 2π additions. Note that additions of 2π accumulate [see Eqs. (1)–(3) in Ref. [1]]. It is easy to see that this procedure provides 2π phase increase for each revolution only if the points in the phase plane lay on a concave curve. Otherwise, the increase of the phase is larger; moreover, its exact value depends on the number of the points on the convex part of the curve. In the example shown in Fig. 1, ψ increases by 4π at one revolution. Hence, the frequency obtained from the "phase" variable ψ is not correct.

Counterexample 2. Consider a discrete scalar signal $s_i=A\sin\omega\tau i$. Then, according to Ref. [1],

$$\phi_{i+1}=\arctan\frac{\{A\sin[\omega\tau\cdot(i+1)]-A\sin(\omega\tau\cdot i)\}}{\tau}.$$

Obviously, this expression does not provide a correct phase because (i) generally $\phi_{i+1}-\phi_i\neq\omega\tau$, i.e., the "phase" is not a linearly growing function, (ii) it depends on the amplitude, and (iii) recomputation $\phi_i\rightarrow\psi_i$ makes the phase increase within one period of the sine wave larger than 2π (the exact value depends on $\omega\tau$).

We emphasize that only continuous-time self-sustained oscillators have an ability to synchronize their phases and frequencies due to a weak forcing or coupling (what is usually termed as "phase locking" in the context of regular oscillations and "phase synchronization" in the context of chaotic ones). Defined according to Ref. [4], the phase of such systems corresponds to the translation of the point in the phase space along the limit cycle or chaotic trajectory that belongs to the attractor, i.e., the phase parametrizes the position on the zero Lyapunov exponent flow manifold. As this direction is neutrally stable, the phase of the oscillator can be easily adjusted and its frequency can be continuously

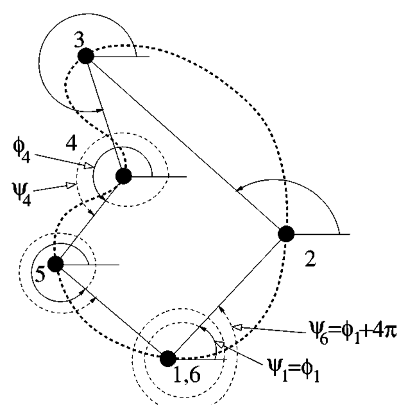

FIG. 1. The phase definition of Chen *et al.* [1] does not provide correct increase of phase within one rotation along the curve that is not everywhere concave. As an example, a five-periodic motion is considered; points 1 and 6 coincide. The algorithm of Ref. [1] gives $\psi_6=\psi_1+4\pi$. For each point, the auxiliary variable ϕ_i is shown by an arrowed solid arc. The "phase" variable ψ_i (if it differs from ϕ_i) is shown as a continuation by a dashed arc (cf. Fig. 1 in Ref. [1]).

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changed [5]. Generically, maps do not have zero Lyapunov exponents and, hence, the notion of phase for these systems is doubtful. If a map has a periodic orbit, its discrete period cannot be changed gradually.

To conclude, the angle variable ψ proposed by Chen *et al.* [1] does not satisfy basic properties of a phase. Hence, their results cannot be considered in the framework of phase synchronization. Not every angle variable is phase.

[1] J. Y. Chen *et al.*, Phys. Rev. E **61**, 2559 (2000).

[2] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[3] A. S. Pikovsky, M. G. Rosenblum, G. V. Osipov, and J. Kurths, Physica D **104**, 219 (1997).

[4] Suppose a chaotic oscillator admits definition of a Poincaré secant surface. Then, for each piece of a trajectory between two cross sections with this surface the phase is defined as a linear function of time, so that the phase increment is 2π at each rotation [3]: $\varphi(t)=2\pi(t-t_n)/(t_{n+1}-t_n)+2\pi n$, $t_n \leqslant t < t_{n+1}$. Here t_n is the time of the nth crossing of the secant surface.

[5] We note that nonlinear oscillators that exhibit chaos due to periodic driving (e.g., periodically forced Duffing oscillator in a chaotic regime) do not have a zero Lyapunov and cannot be synchronized. The phase of such nonautonomous systems is predetermined by the phase of the drive and cannot be adjusted by another force or due to interaction with another system.